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Absence of Screening in Certain Lattice Gauge and Plasma Models

D. C. Brydges[1] **and E. Seiler**[2,3]

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We discuss some Abelian lattice gauge models of the noncompact variety, including models of relativistic and nonrelativistic plasmas. For all these models we show absence of exponential clustering for some observables in some domains in parameter space. We comment on the physical meaning of these results, in particular with respect to Debye screening of static electric charges.

KEY WORDS: Coulomb systems; Debye screening; Higgs transition; plasmas.

1. INTRODUCTION

One of the major problems in gauge theories is to understand the so-called Higgs transition in models with matter. This transition is supposed to have had a major effect on the development of the early universe and it is also occurring in superconductors. But unlike the magnetizing transition of ferromagnetic spin systems which is a limiting case of it, not much rigorous understanding of it been achieved.

In Ref. 1 it was shown that in three space-time dimensions Abelian models of the so-called noncompact type show such a transition. In Ref. 2 it was shown that such models always have a Higgs phase characterized by exponential decay of correlations and exponential screening of both electric and magnetic fields.

Here we prove that the same type of models always has a massless phase with Coulombic behavior of electric and magnetic fields and a

[1] Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903.
[2] Department of Physics, Princeton University, Princeton, New Jersey 08544.
[3] Permanent address: Max-Planck-Institut für Physik und Astrophysik, Postfach 401212, D-800 München, Federal Republic of Germany.

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massless photon, provided the dimension of space-time is at least 3 and the matter fields are sufficiently massive. Our proof is more elementary and more general than the argument given in Ref. 1; it allows one to treat fermionic as well as bosonic matter.

By the same technique we also prove some results for certain models of a plasma: First we show that a relativistic plasma described by a finite temperature lattice gauge model of the type considered before, has a plasma phase characterized by the absence of screening of static magnetic fields, aside from a possible Higgs or superconducting phase that screens all fields. Second we interpret a nonrelativistic quantum Coulomb system (on a lattice) as a kind of lattice gauge theory and find that the correlation length for nonstatic observables is infinite.

Since in both plasma models static electric sources couple to the long-range fields by way of the matter we are led to conjecture: *Neither in a relativistic nor in a nonrelativistic plasma Debye screening holds in the strict sense*; by this we mean that the electric field of a static external source is not screened exponentially but has a long-range tail decaying like an inverse power of the distance. This type of conjecture has previously appeared in Ref. 10b, p. 428. Of course we do not want to suggest that the Gaussian approximation that predicts exponential screening[3] is not very close to the truth for all but extremely large distances.

2. A GENERAL RESULT

Here we want to show that massless behavior is a consequence of the general structure of the effective interaction of the gauge field arising from integrating out the matter fields, provided they are massive enough, keeping the vecuum polarization effects small. The proof is obviously inspired by the method of Fröhlich and Spencer[4] to prove the absence of screening in dipole gases (see also Park[5]). In fact the reason can be stated in short as followed: Vacuum polarization produces (electric and magnetic) dipoles; dipoles do not screen.

We consider the following class of models:

The electromagnetic field is described by a real value 1-form A on a lattice $\mathbb{Z}^\nu$ or $\Lambda c\mathbb{Z}^3$, $\nu \geqslant 3$. (We use the familiar formalism of chains, cochains = forms, exterior derivatives etc. on the lattice. See, for instance, Ref. 6).

We want to study Gibbs measures of the form

$$\frac{1}{Z_\Lambda} e^{-S_\Lambda} \prod_{l \in \Lambda^{(1)}} dA_l \tag{2.1}$$

and their thermodynamic limit(s) $\Lambda \to \mathbb{Z}^\nu$. $\Lambda^{(1)}$ denotes the set of links (1-cells) in Λ. The action S_Λ is

$$S_\Lambda = S_{\text{Y.M.},\Lambda} + S_{M,\Lambda} \tag{2.2}$$

where

$$S_{\text{Y.M.},\Lambda} = \frac{1}{2e^2}\left[(dA, dA)_\Lambda + \alpha(d^*A, d^*A)_\Lambda\right] \tag{2.3}$$

$\alpha = 1$ is the Feynman gauge, $\alpha = \infty$ the Landau gauge. For convenience we will set $\alpha = 1$ from now on.

S_M arises from integration over some matter fields and possibly some of the degrees of freedom of the gauge field. Here we just assume

$$-S_{M,\Lambda} = \operatorname{Re} \sum_{\substack{j \\ \operatorname{supp} j \subset \Lambda}} c_j e^{i(A,j)} \tag{2.4}$$

where the sum is over a set of 1-chains (which we identify with 1-forms) j corresponding to closed loops in the finite lattice $\Lambda \subset \mathbb{Z}^\nu$. This means that the j's are coclosed ($d^*j = 0$) and integer valued. The coefficients c_j are real valued and approximately independent of Λ and translation invariant. From the structure of the proof it will become clear what kind of violation of Λ independence of the c_j can be tolerated; for suitable (e.g., free) boundary conditions they will be exactly independent of Λ and translation invariant. We will not discuss this question any further.

Finally we have to make a convergence assumption:

(C) There is an $\varepsilon > 0$, independent of Λ, $l \in \Lambda^{(1)}$ (the set of links in Λ) such that

$$\sum_{l \in \operatorname{supp} j} |C_j|^{\varepsilon \|j\|_1} < \infty \tag{2.5}$$

where

$$\|j\|_1 = \sum_l |j_l| \tag{2.6}$$

Translation invariant thermodynamic limits of the measures (2.1) always exist by compactness arguments. In Theorem 2.1 $\langle\,.\,\rangle$ denotes such a limit or a finite volume expectation.

The result we want to prove in this section is as follows:

Theorem 2.1. Let j_c be a real-valued, coclosed 1-form of compact support. Then under the condition (C)

$$\langle (j_c, A)^2 \rangle \geqslant \bar{e}^2 (j_c, -\Delta^{-1} j_c)$$

where $\bar{e}^2 > 0$ is independent of Λ and $-\Delta = d^*d + dd^*$ is the Laplacian for $\mathbb{Z}^\nu$ or for Λ, respectively.

This has the following consequence:

Corollary 2.2. In any translation invariant thermodynamic limit of (2.1) the two-point function of dA does not decay exponentially.

Proof of the Corollary. By the theorem the Fourier transform of the two-point function has a singularity at the origin. So by the Paley–Wiener theorem the two-point function cannot decay exponentially. ▮

Remark. It can also be seen, using an argument of Ref. 4, that the two-point function of the field strength dA is not absolutely summable: Its Fourier transform is not continuous (though bounded) at the origin, so it is not the Fourier transform of a finite measure.

Proof of the Theorem. We use the complex translation method of Refs. 7 and 4.

By Cauchy's theorem we may replace A by $A + iB$ in the unnormalized expectation

$$[e^{i(j_c, A)}]_\Lambda \equiv Z_\Lambda \langle e^{i(j_c, A)} \rangle_\Lambda = \int e^{i(j_c, A) - S_\Lambda(A)} \tag{2.7}$$

[provided we do not leave the domain of convergence of the series (2.4)]. B is again a real-valued 1-chain (supported in Λ). Working out the exponent of the integrand in (2.7) we obtain

$$\begin{aligned} &i(j_c, A) - S_M(A) - S_{\text{Y.M.},\Lambda}(A) \\ &\quad - (j_c, B) + \sum c_j[\cos(A + iB, j) - \cos(A, j)] \\ &\quad + \frac{1}{2e^2}[(dB, dB) + (d^*B, d^*B)] \\ &\quad - \frac{i}{e^2}[(dA, dB) + (d^*A, d^*B)] \end{aligned} \tag{2.8}$$

This leads to the estimate

$$|[e^{i(j_c, A)}]|_\Lambda \leqslant \int e^{-S_\Lambda + F}$$

where

$$F = \frac{1}{e^2}[(dB, dB) + (d^*B, d^*B)] + \sum_j c_j[\cosh(B, j) - 1] - (B, j_c) \tag{2.9}$$

$$\log|\langle e^{i(j_c, A)} \rangle_\Lambda| \leqslant \frac{1}{2e^2}(B, -\Delta B) + \sum_j c_j[\cosh(B, j) - 1] - (B, j_c) \tag{2.10}$$

In (2.10) we can take the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^\nu$. We should then optimize the bound by choosing a good B. Making an educated guess (educated by reading Ref. 4) we choose the potential appropriate to a current loop in a polarizable medium,

$$B = -\gamma \Delta^{-1} j_c$$

where Δ^{-1} is the inverse of the infinite volume Laplacian (the fact that this B does not have compact support clearly does not cause any problems). This gives

$$\log |\langle e^{i(j_c, A)} \rangle| \leqslant (j_c, -\Delta^{-1} j_c) \left(\frac{\gamma^2}{2e^2} - \gamma \right) + \sum_j c_j [\cosh \gamma (j_c, \Delta^{-1} j_c) - 1] \tag{2.11}$$

This inequality becomes an equality for $j_c = 0$. So if we replace j_c by εj_c, divide by ε^2, and take the limit $\varepsilon \to 0$ [there are no $O(\varepsilon)$ terms] we obtain

$$\frac{1}{2} \langle (j_c, A)^2 \rangle \geqslant (j_c, -\Delta^{-1} j_c) \left(-\frac{\gamma^2}{2e^2} + \gamma \right) - \frac{\gamma^2}{2} \sum_j c_j (j, \Delta^{-1} j_c)^2 \tag{2.12}$$

This limiting procedure is easy to justify using condition (C).

Now we look at the sum over j more closely. We claim the following:

Lemma 2.3. Under the condition (C)

$$\left| \sum_j c_j (j, \Delta^{-1} j_c)^2 \right| \leqslant Kc(j_c, -\Delta^{-1} j_c)$$

where c is a geometric constant and

$$K = \sup_l \sum_{\operatorname{supp} j \ni l} |c_j| \, \|j\|_1^6 \tag{2.13}$$

Proof. We "decompose j into loops":

$$j = \sum_{k=1}^{\|j\|_\infty} j_k$$

where j_k takes only the values 0, ± 1 and $d^* j_k = 0$ for all k. We require that each j_k describes a closed loop and for any given link all the loops going through l go through it in the same direction.

We may "span a minimal surface" into each loop, i.e., we may write

$$j_k = d^* \omega_k$$

where ω_k are integer-valued 2-forms obeying

$$\|\omega_k\|_1 \leqslant c \|j_k\|_1^2 \tag{2.14}$$

with some universal constant c.

We prove (2.14) with $c = 1/2$:

Lemma 2.4. Let j be a 1-chain describing a closed loop. Then there is a 2-form $\omega(j)$ describing a surface bordered by j such that

$$\|\omega(j)\|_1 \leqslant \tfrac{1}{2} \|j\|_1^2$$

Proof. We proceed by induction on $\|j\|_1$. For $\|j\|_1 \leqslant 4$ the lemma obviously holds. Let now $\|j\|_1 \geqslant 4$. Let

$$E_t \equiv \{x \in \mathbb{Z}^d \mid x_d = t\} \tag{2.15}$$

$$T \equiv \sup\{t \mid \operatorname{supp} j \cap E_t \neq 0\} \tag{2.16}$$

By possibly reordering the coordinates of $\mathbb{Z}^d$ we may assume that $\operatorname{supp} j \not\subset E_T$. Let $j^{(T)}$ be the part of j lying in E_T, $j^{(T-1)}$ the translate of $j^{(T)}$ lying in E_{T-1},

$$d^* j^{(T)} \equiv \rho^{(T)}, \qquad d^* j^{(T-1)} \equiv \rho^{(T-1)} \tag{2.17}$$

l a 1-chain satisfying

$$d^* l = \rho^{(T)} - \rho^{(T-1)}, \qquad \|l\|_1 = \|\rho^{(T)}\|_1$$

(i.e., l consists of the links in the d direction connecting $\rho^{(T)}$ and $\rho^{(T-1)}$. Define

$$j_1 \equiv j^{(T)} - j^{(T-1)} - l, \qquad j_2 \equiv j - j_1 \tag{2.18}$$

Then $d^* j_1 = 0 = d^* j_2$ and

$$\|j_2\| \leqslant \|j\|_1 - \|l\|_1, \qquad \|j^{(T)}\|_1 \leqslant \|j\|_1 - \|l\|_1 \tag{2.19}$$

$$\|l\|_1 \geqslant 2 \tag{2.20}$$

It is obvious how to define $\omega(j_1)$ such that

$$\|\omega(j_1)\|_1 = \|j^{(T)}\|_1 \tag{2.21}$$

[$\omega(j_1)$ consists of the plaquettes determined by the links of $j^{(T)}$ and their translates in $j^{(T-1)}$].

By the induction hypothesis there is a $\omega(j_2)$ with $d^*\omega(j_2)=j_2$, taking values 0, ± 1, such that

$$\|\omega(j_2)\|_1 \leqslant \tfrac{1}{2}\|j_2\|_1^2 \tag{2.22}$$

Then define $\omega(j) \equiv \omega(j_1)+\omega(j_2)$. It obeys by construction

$$\|\omega(j)\|_1 \leqslant \|j^{(T)}\|_1 + \tfrac{1}{2}\|j_2\|_1^2 \tag{2.23}$$

By (2.19), (2.20) this is bounded by

$$\|j\|_1 - 2 + \tfrac{1}{2}(\|j\|_1 - 2)^2 \leqslant \tfrac{1}{2}\|j\|_1^2 \tag{2.24}$$

as desired. ∎

$\sum_k \omega(j_k) \equiv \omega(j)$ is an integer-valued 2-form satisfying

$$\|\omega(j)\|_1 \leqslant \sum_k \|\omega_k(j)\|_1 \leqslant \sum_k c\,\|j_k\|_1^2 \leqslant c\,\|j\|_1^2 \tag{2.25}$$

So we can write

$$\begin{aligned}\left|\sum_j c_j(j_c, \Delta^{-1}j)^2\right| &= \left|\sum_j c_j(j_c, \Delta^{-1}d^*\omega(j))^2\right| \\ &= \Bigg|\sum_j c_j \sum_{P,P' \in \operatorname{supp}\omega(j)} (j_c, \Delta^{-1}\, d^*\omega_p) \\ &\quad \times (j_c, \Delta^{-1}\, d^*\omega_p)(\omega(j), p)(\omega(j), p')\Bigg| \end{aligned} \tag{2.26}$$

where ω_p is the unit 2-form associated with the plaquette p and we understand by $p \in \operatorname{supp}\omega$ that $(\omega, p) > 0$.

The last expression ccan be estimated (using Schwarz's inequality) followed by $\|\omega(j)\|_2^2 \leqslant \|\omega(j)\|_1^2$ by

$$\sum_j |c_j|\,\|\omega(j)\|_1^2 \sum_{P \in \operatorname{supp} j} (dj_{c'}\Delta^{-1}\omega_p)^2 \tag{2.27}$$

So

$$\left|\sum_j c_j(j_c, \Delta^{-1}j)^2\right| \leqslant \sum_P (dj_c, \Delta^{-1}\omega_p)^2 \sum_{j:\, P \in \operatorname{supp}\omega(j)} |c_j|\,\|\omega(j)\|_1^2 \tag{2.28}$$

Using

$$\sum_p (dj_c, \Delta^{-1}\omega_p)^2 = (j_c, \Delta^{-2}\, dj_c) = (j_c, -\Delta^{-1} j_c) \tag{2.29}$$

and the bound

$$\sum_{j:\, P \in \operatorname{supp} \omega(j)} |c_j| \, \|\omega(j)\|_1^2 \leqslant \sum_{j:\, l \in \operatorname{supp} j} |c_j| \, \|\omega(j)\|_1^3$$

(because there exists a link within distance $\|\omega(j)\|_1$ of P through which j passes)

$$\leqslant \sum_{j:\, \operatorname{supp} j \ni l} |c_j| \, c \, \|j\|_1^6 = K \cdot c \tag{2.30}$$

by (2.25) we conclude

$$\left| \sum c_j (j_c, \Delta^{-1} j)^2 \right| \leqslant K \cdot c (j_c, -\Delta^{-1} j_c) \quad \blacksquare$$

Inserting the lemma into (2.12) gives

$$\frac{1}{2} \langle (j_c, A)^2 \rangle \geqslant (j_c, -\Delta^{-1} j_c) \left[-\frac{\gamma^2}{2} \left(\frac{1}{e^2} + Kc \right) + \gamma \right] \tag{2.31}$$

This bound becomes optimal for

$$\gamma = 2\bar{e}^2 \equiv (e^{-2} + Kc)^{-1} \tag{2.32}$$

and gives with this choice

$$\langle (j_c, A)^2 \rangle \geqslant \bar{e}^2 (j_c, -\Delta^{-1} j_c)$$

which concludes the proof of the theorem. $\blacksquare$

We can see from this result that occurrence of the Higgs mechanism is necessarily accompanied by a breakdown of the convergence condition (C). A familiar example is furnished by the Schwinger model where (in the continuum)

$$S_M = -\frac{e^2}{2\pi} (dA, -\Delta^{-1} dA) \tag{2.33}$$

If we interpret (2.33) as a lattice action it clearly does not have an expansion of the form (2.4) satisfying condition (C).

3. ZERO TEMPERATURE LATTICE GAUGE MODELS WITH MATTER

These models are defined in terms of an action

$$S = S_{\text{Y.M.}} + \tilde{S}_M \tag{3.1}$$

where $M = H, F$ refers to either scalar (H) or fermionic (F) matter fields.

A Higgs field ϕ is a complex-valued scalar field living on the sites of the lattice:

$$\tilde{S}_{H,\Lambda} = \kappa \operatorname{Re} \sum_{\langle xy \rangle \in \Lambda^{(1)}} \phi(x)\, e^{iA_{xy}}\, \phi(y) + \sum_{x \in \Lambda} V(|\phi(x)|) \tag{3.2}$$

The potential V has to satisfy a bound

$$V(R) \geqslant cR^2$$

and is sometimes replaced by the constraint $|\phi(x)| = R_0$.

The fermion fields ψ, $\bar{\psi}$ are generators of a Grassmann algebra living on the sites of the lattice and

$$\tilde{S}_F = \kappa \sum_x \bar{\psi}(x) \Gamma_{xy} e^{iAxy}\, \psi(y) + \textstyle\sum_x \bar{\psi}(x)\, \Gamma \psi(x) \tag{3.3}$$

Here Γ_{xy}, Γ are some finite-dimensional matrices. The Gibbs measure for scalar matter is

$$\frac{1}{Z_\Lambda} e^{-S_{\mathrm{Y.M.},\Lambda} - \tilde{S}_{\mathrm{H},\Lambda}} \prod_l dA_l \prod_x d\phi(x)\, d\overline{\phi(x)} \tag{3.4}$$

whereas for fermions the role of the Gibbs measure is played by the "Berezin integral"

$$\frac{1}{Z_\Lambda} e^{-S_{\mathrm{Y.M.}} - \tilde{S}_F} \prod_x d\psi(x)\, d\bar{\psi}(x)$$

For details see, for instance, Ref. 8.

We define

$$-S_{M,\Lambda} \equiv \log \int e^{-\tilde{S}_{M,\Lambda}}\, d(\text{matter}) \tag{3.5}$$

It is then a standard application of the polymer expansion (see for instance Ref. 8) to show that for sufficiently small κ $S_{M,\Lambda}$ is indeed of the form (2.4) and satisfies (C).

It should be remarked that small κ corresponds to very massive matter fields. If we use the constraint $|\phi(x)| = R_0$, κR_0^2 is the relevant parameter that has to be small.

We thus obtain the following.

Theorem 3.1. For bosonic or fermionic matter there is no mass gap (exponential decay of correlations) provided the hopping parameter κ is sufficiently small or, equivalently, the mass of the matter fields is sufficiently large.

4. FINITE TEMPERATURE ABELIAN LATTICE GAUGE MODELS (RELATIVISTIC PLASMAS)

We consider the same kind of models as in the previous section. But now we go to finite temperature, that is, we keep the lattice finite and periodic in time direction with period β/τ (τ is the lattice spacing in time direction). For fermion fields antiperiodic boundary conditions have to be used.

The gauge field can be decomposed into frequency modes:

$$A_{xy}(t) = \sum_{n=0}^{\beta/\tau - 1} \cos \frac{2\pi n t}{\beta} A_{xy}^{(n)} \tag{4.1}$$

(The time t runs through $0, \tau, 2\tau, ..., \beta - \tau$).

We are interested mostly in the "static magnetic mode" $\mathbf{A}_{\mathbf{xy}}^{(0)}$ where $\langle \mathbf{xy} \rangle$ runs through the spatial links.

Keeping at first a finite spatial lattice Λ_0 we define as before

$$-S_{M,\Lambda_0}\langle A \rangle \equiv \log \int e^{-\tilde{S}_M} d(\text{matter}) \tag{4.2}$$

Again the polymer expansion can be used to see that (2.4) and (C) hold.

There is one difference, however: The sum over loops (2.4) now also contains noncontractible loops (so-called Polyakov loops).

In a Gaussian approximation those Polyakov loops seem to give a mass to the "static electric mode" $A_{xy}^{(0)}$ where $\langle xy \rangle$ is in time direction.

We want to study the static magnetic properties of the system, however. For this purpose we introduce a static external curent j_c, i.e., j_c is a coclosed, real-valued, time independent 1-form supported on spatial links.

We follow the patern of the proof of Theorem 2.1 to estimate

$$[e^{i(j_c, A)}]_{\Lambda_0}$$

There might seem to be a problem because $\Delta = dd^* + d^*d$ now does not have an inverse since there are harmonic 1-forms due to the periodic boundary conditions in time. Our static external current j_c is, however, orthogonal to the kernel of Δ and we may define unambiguously

$$B = -\gamma \Delta^{-1} j_c$$

as before. In fact we may interpret Δ in this formula as the spatial Laplacian since j_c is static.

Proceeding now as in the proof of Theorem 2.1 we obtain the following:

Theorem 4.1. In a relativistic plasma as defined in this section static magnetic fields are not screened provided the matter fields are sufficiently massive (i.e., the hopping parameter is small enough).

It should be remarked that by the methods of Ref. 2 the existence of a Higgs phase for the bosonic model can be seen also at finite temperature. In this phase magnetic fields are screened exponentially. It should be interpreted as a model of a superconductor.

There is also an example for a Higgs phase of a fermionic plasma: This is again the Schwinger model, this time at finite temperature. Whereas in the bosonic model increasing the temperature tends to destroy the Higgs mechanism this is not so in the Schwinger model.

5. NONRELATIVISTIC QUANTUM COULOMB SYSTEMS (NONRELATIVISTIC PLASMA)

It has been firmly established some time ago that classical Coulomb systems have a plasma phase showing Debye screening.[9,10]

There is a difficulty, however, if one tries to extend this result to the quantum statistical setting, which has been noticed long ago (P. Federbush, private communication to D. C. Brydges; see also Ref. 10b): While it seems that a mass gap is generated at lowest order for static correlations this is not so for nonstatic ones. The method described in Section 2 can be used to give a proof that there is no screening for nonstatic sources (of a certain kind).

To apply our method it is convenient (and maybe of some interest) to cast the quantum Coulomb system into the form of some kind of lattice gauge model (cf. Refs. 10b and 17).

The Hamiltonian for n positive and n negative particles on the lattice $\Lambda_0 \subset \mathbb{Z}^d$ is

$$H_{2n} = -\frac{1}{2M}\Delta_{2n} - e^2 \sum_{i,j} V_c(x_i - y_j) + \frac{e^2}{2} \sum_{i,j} [V_c(x_i - x_j) + V_c(y_i - y_j)] \tag{5.1}$$

Δ is the lattice Laplacian, Δ_{2n} the corresponding operator on the $2n$-particle space.

For Boltzmann statistics H_{2n} is considered as an operator on $\bigotimes^{2n}(l^2(\Lambda_0)) \equiv \mathscr{H}_{2n,B}$.

For Bose statistics H_{2n} acts on

$$\mathscr{H}_{2n,b} \equiv \bigotimes_{s}^{2n} (l^2(\Lambda_0))$$

and for Fermi statistics on

$$\mathscr{H}_{2n,f} \equiv \Lambda^{2n}(l^2(\Lambda_0))$$

The grand canonical partition function for the neutral system is

$$\Xi_B = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2} \operatorname{Tr}_{\mathscr{H}_{2n,B}} e^{-\beta H_{2n}} \qquad \text{(Boltzmann)} \tag{5.2}$$

$$\Xi_s = \sum_{n=0}^{\infty} z^{2n} \operatorname{Tr}_{\mathscr{H}_{2n,s}} e^{-\beta H_{2n}} \qquad (s = \text{Bose or Fermi}) \tag{5.3}$$

We now introduce the well-known sine-Gordon (Siegert) transformation[16]: Let $d\mu(A_0)$ be the Gaussian measure with mean zero and covariance

$$\langle A_0(x, t)\, A_0(x', t')\rangle = V_c(x - x')\, \delta(t - t') \tag{5.4}$$

where $V_c(x - x') = (-\Delta^{-1})(x - x')$ is the lattice Coulomb potential. Then

$$\operatorname{Tr}_{\mathscr{H}_{2n,B}} e^{-\beta H} = \int d\mu_c(A_0) \left| \operatorname{Tr}_{\mathscr{H}_1} T \exp\left[-\int_0^\beta (H_0 + ieA_0)\, d\tau \right] \right|^{2n} \tag{5.5}$$

and

$$\operatorname{Tr}_{\mathscr{H}_{2n,s}} e^{-\beta H} = \int d\mu_c(A_0) \operatorname{Tr}_{\mathscr{H}_{2n,s}} T \left(\left\{ \exp\left[-\int_0^\beta (H_0 + ieA_0)\, d\tau \right] \right\}^{\otimes n} \right.$$
$$\left. \times \left\{ \exp\left[-\int_0^\beta (H_0 - ieA_0)\, d\tau \right] \right\}^{\otimes n} \right) \tag{5.6}$$

Here H_0 is the free one-particle Hamiltonian $-(1/2M)\Delta$ and T denotes time ordering. To verify these formulas it is easiest to expand in powers of e and use Wick's theorem.

(5.5) and (5.6) contain only one-particle Hamiltonians. (5.5) can be summed directly to give the grand canonical partition function

$$\Xi_B = \int d\mu(A_0) \exp\left\{ 2 \operatorname{Re} z \operatorname{tr}_{\mathscr{H}_1} T \exp\left[-\int_0^\beta (H_0 + ieA_0)\, d\tau \right] \right\} \tag{5.7}$$

For Bose and Fermi statistics we use the standard formula for an uncoupled system

$$\begin{aligned}\log \Xi_s &= -\mathrm{tr}_{\mathscr{H}_1} \log(1 - z\varepsilon_s e^{-\beta H}) \\ &= \sum_{n=1}^{\infty} \frac{(z\varepsilon_s)^n}{n} \mathrm{tr}_{\mathscr{H}_1} e^{-\beta H}\end{aligned} \tag{5.8}$$

where H is a one-particle Hamiltonian and

$$\varepsilon_s = \begin{cases} 1 & \text{(Bose)} \\ -1 & \text{(Fermi)} \end{cases}$$

This gives for the Coulomb system

$$\Xi_s = \int d\mu(A_0) \exp \sum_{n=1}^{\infty} \frac{(z\varepsilon_s)^n}{n} \mathrm{tr}_{\mathscr{H}_1} T \exp\left[-\int_0^{n\beta} (H_0 + ieA_0)\, d\tau \right] \tag{5.10}$$

where A_0 is extended periodically with period β. (5.7) and (5.10) can be rewritten using a lattice Feynman–Kac formula. Let $dP^t_{xy}(\omega)$ be the "lattice Brownian bridge" for paths starting at x and ending at y after t. It is related to the lattice heat kernel in the same way as the Brownian bridge to the usual heat kernel:

$$(\exp t\Delta)(x, y) = \int dP^t_{xy}(\omega) \tag{5.11}$$

The paths w have to be considered as functions

$$\omega\colon [0, t] \to \Lambda_0 \tag{5.12}$$

where Λ_0 is the spatial lattice.

This allows us to write

$$\mathrm{tr}_{\mathscr{H}_1} \exp\left[-\int_0^{\beta} (H_0 + ieA_0)\, d\tau \right] = \int dP^{\beta/2M}_{xx}(\omega)\, e^{ie(A_0, \omega)} \tag{5.13}$$

where we write (A_0, ω) for

$$\int_0^{\beta} A_0(\omega(\tau), \tau)\, d\tau$$

to bring out the analogy with (2.4). Thus ω is identified with a charge density concentrated on the path ω.

Inserting (5.13) in (5.7) and (5.10) we obtain

$$\Xi_B = \int d\mu(A_0)\, e^{-S_M(A_0)} \tag{5.14}$$

where

$$-S_{M,\beta}(A_0) = z \sum_x 2 \int dP_{xx}^{\beta/2M}(\omega) \cos(A_0, \omega) \tag{5.15}$$

$$-S_{M,s}(A_0) = \sum_x \sum_{n=1}^{\infty} \frac{(\varepsilon_s z)^n}{n} 2 \int dP_{xx}^{n\beta/2M}(\omega) \cos(A_0, \omega) \qquad \text{(Base or Fermi)} \tag{5.16}$$

The similarity to (2.40) is now apparent. The main difference is that the nonrelativistic system has no spatial gauge potentials; furthermore the loops have to run either in the positive or the negative time direction without backtracking (this corresponds to the absence of pair creation).

In fact it is possible to derive (5.16) as a limit of a relativistic lattice gauge model with continuous time; this requires one to introduce two different matter fields for the positive and negative particles and to give the first one a chemical potential $+\mu$ and the second one $-\mu$ coupled to the charge. To get a finite limit as the speed of light $c \to \infty$, μ has to be chosen of the order Mc^2 where M is the particle mass.

As in the previous section we now decompose A_0 into frequencies by Fourier transforming in the time direction.

$$A_0(x, t) = \phi_0(x) + \sum_n \sqrt{2} \cos \frac{2\pi nt}{\beta}\, C_n(x) + \sum_{n=0}^{\infty} \sqrt{2} \sin \frac{2\pi nt}{\beta}\, S_n(x) \tag{5.17}$$

The measure $d\mu(A_0)$ factorizes:

$$d\mu(A_0) = \prod_{n=1}^{\infty} dv(C_n) \prod_{n=1}^{\infty} dv(S_n)\, dv(\phi_0) \tag{5.18}$$

where each dv is Gaussian with mean zero and covariance

$$\beta V_c(x - x')$$

We now introduce an external source $\rho(x)$ (a function on the lattice with compact support) and consider

$$\langle (C_m, \rho)^2 \rangle, \qquad \langle (S_m, \rho)^2 \rangle$$

The main result of this section is as follows:

Theorem 5.1. Let M be sufficiently large and in the case of Bose or Fermi statistics also z sufficiently small. Then for $m \neq 0$

$$\langle (C_m, \rho)^2 \rangle_{\Lambda_0} \geqslant c(M, z, \beta)(\rho, -\Delta^{-1}\rho) \tag{5.19}$$

$$\langle (Sm, \rho)^2 \rangle_{\Lambda_0} \geqslant c(M, z, \beta)(\rho, -\Delta^{-1}\rho(\tag{5.20}$$

where $c(M, z, \beta) > 0$ is independent of Λ_0.

Proof. We discuss the case of Boltzmann statistics in detail and comment on the adaptation of the proof of Bose/Fermi statistics.

In order to analyze the "Polyakov loop" $(A_0, \omega) = \int_0^\beta A_0(\omega(\tau), \tau)\, d\tau$ it is convenient to split the path ω, considered as a 1-form, into two parts

$$\omega = \omega_0 + \omega_1 \tag{5.21}$$

where ω_0 is the straight path satisfying

$$\omega_0(\tau) = \omega(0) \qquad \text{(for all } \tau) \tag{5.22}$$

ω_1 is then a contractible loop intersecting each time slice twice: once going up along ω and once going down along $-\omega_0$.

We have

$$(A_0, \omega) = (A_0, \omega_0) + (A_0, \omega_1) = (\phi_0, \omega_0) + (A_0 - \phi_0, \omega_1) \tag{5.23}$$

We use again complex translations, but this time we shift only C_m or S_m. Let us consider C_m:

$$C_m(x) \to C_m(x) + iB(x) \tag{5.24}$$

As before we obtain

$$\log |\langle e^{i(C_m, \rho)} \rangle_{\Lambda_0}| \leqslant \frac{1}{2}(dB, dB)_{\Lambda_0} + z \sum_{x \in \Lambda_0} \int dP_{xx}^{\beta/2M}(\omega)[\cosh e(B, \omega_1) - 1] - (\rho, C_m) \tag{5.25}$$

Again we choose $B = -\gamma \Delta^{-1}\rho$ (but now Δ is the spatial Laplacean) and obtain, collecting the terms of second order in ρ:

$$\langle (C_m, \rho)^2 \rangle \geqslant (\rho, -\Delta^{-1}\rho)\left(\gamma - \frac{\gamma^2}{2}\right) - \frac{z\gamma^2}{2} \sum_{x \in \Lambda_0} \int dP^{\beta/2M}(\omega)\, 2e^2 \left(\rho \cos\frac{2\pi m\tau}{\beta}, \Delta^{-1}\omega_1\right)^2 \tag{5.26}$$

The last term needs a little attention. We compute

$$-\left(\rho\cos\frac{2\pi m\tau}{\beta}, \Delta^{-1}\omega_1\right)=\int_0^\beta d\tau\sum_{z,y}\rho(z)\cos\frac{2\pi m\tau}{\beta}\,\omega_1(y,\tau)\,V_c(z-y) \quad (5.27)$$

Since

$$\omega_1(y,\tau)=\delta_{y\omega(\tau)}-\delta_{y\omega(0)} \quad (5.28)$$

this gives

$$-\left(\rho\cos\frac{2\pi m\sigma}{\beta}, \Delta^{-1}\omega_1\right)$$
$$=\int_0^\beta d\tau\sum_z\rho(z)\cos\frac{2\pi m\sigma}{\beta}\,\{V_c[z-\omega(\tau)]-V_c[z-\omega(0)]\} \quad (5.29)$$

Therefore by Cauchy–Schwarz in the τ variable

$$X\equiv\sum_x 2\int dP_{xx}^{\beta/2M}(\omega)\left(\rho\cos\frac{2\pi m\tau}{\beta}, \Delta^{-1}\omega_1\right)^2$$
$$\leqslant\beta\int dP_{00}^{\beta/2M}\int_0^\beta d\tau\sum_{x,z,z'}2\rho(z)\,\rho(z')\left(\cos\frac{2\pi m\tau}{\beta}\right)^2$$
$$\times\;\{V_c[z-\omega(\tau)-x]-V_c(z-x)\}\{V_c[z'-\omega(\tau)-x]-V_c(z'-x)\}$$
$$(5.30)$$

The convolution in the last expression can be carried out and gives

$$X\leqslant\beta\int dP_{00}^{\beta/2M}(\omega)\int_0^\beta d\tau\sum_{z,z'}2\rho(z)\,\rho(z')\left(\cos\frac{2\pi m\tau}{\beta}\right)^2$$
$$\times\;\{2\Delta^{-2}(z-z')-2\Delta^{-2}[z-z'-\omega(\tau)]\} \quad (5.31)$$

[Note that even though Δ^{-2} is not well defined for $v\leqslant 4$, the difference appearing in (5.31) is.]

This is easiest to estimate in Fourier space:

$$X\leqslant\beta\int dP_{00}^{\beta/2M}(\omega)\int_0^\beta d\tau\int d^dk\,2\,|\rho(k)|^2$$
$$\times\;\{2-2\cos[k\cdot\omega(\tau)]\}\left[\sum_{j=1}^d(2-2\cos k_j)\right]^{-2}$$
$$\leqslant K'\beta\int dP_{00}^{\beta/2M}(\omega)\int_0^\beta d\tau\,\omega(\tau)^2\int d^dk\left[\sum_{j=1}^d(2-2\cos k_j)\right]^{-1}$$
$$=K_\beta(\rho,-\Delta^{-1}\rho) \quad (5.32)$$

where

$$K_\beta = K'\beta \int dP_{00}^{\beta/2m}(\omega) \int_0^\beta d\tau\omega(\tau)^2$$
$$\leqslant K'(\beta^3/2M)\,\text{const} \tag{5.33}$$

$$K' = 2 \sup_a \frac{1}{a^2}(1 - \cos k\cdot a)\left[\sum_{j=1}^{d}(1-\cos k_j)\right]^{-1} \tag{5.34}$$

Inserting this bound into (5.26) we obtain

$$\langle (C_m, \rho)^2 \rangle \geqslant (\rho, -\Delta^{-1}\rho)\left[\gamma - \frac{\gamma^2}{2}(1 + ze^2K_\beta)\right] \tag{5.35}$$

and choosing γ optimally concludes the proof for Boltzmann statistics.

For Bose and Fermi statistics we need to control the sum over n. This is no problem provided z is small enough.

Note that again the argument does not work for $m=0$ because in Gaussian approximation a mass is generated for ϕ_0 as in Section 4.

One might ask whether the proof could also be given for the continuum. Clearly this can only work if there is stability of matter, i.e., for Fermi statistics. There does not seem to be any fundamental obstacle to extend our proof to this case.

6. CONCLUSIONS AND OPEN QUESTIONS

The results of Section 3 confirm the conventionally accepted picture of the phase structure of Higgs models: There is a QED-like phase with a massless photon that occurs when the Higgs potential has one rather deep minimum at the origin but also when the length of the Higgs field is frozen to a rather small value $R_0 > 0$. If the Higgs potential develops a deep well far away from the origin or if the fixed length R_0 of the Higgs field becomes large, Balaban *et al.*[2] have shown that the "Higgs mechanism operates," i.e., the theory develops a mass gap and shows exponential screening.

In fermionic models it is also gratifying that we obtain a QED phase for large enough fermion mass. The existence of a "Higgs phase" with a positive mass gap has not been established for fermionic models with the notable exception of the Schwinger model.

Likewise the result of Section 4 showing that a relativistic plasma does not screen magnetostatic fields should be gratifying—though not surprising—to plasma physicists (and the taxpayer).

It should be noted, however, that it was crucial to use a noncompact version of lattice gauge theory to obtain these results. Our proofs do not work if we use Wilson's compact version of lattice gauge theories. There is a standard explanation for this: The compact theory can be interpreted as a model containing magnetic monopoles[15,13] that condense and make it behave like a superconductor in which all fields become massive. We suspect that this happens even for arbitrarily small hopping parameter, i.e., arbitrarily large mass of the matter fields, and does not disappear before we reach the continuum limit. It is not certain that in the continuum limit a QED phase reappears.

This underscores again a fact that has been noted a number of times: The qualitative behavior of compact lattice gauge models is often quite different from the presumed behavior of the continuum theory. Where they exist, noncompact models seem to approximate the continuum models much better than compct ones.

For compact lattice QED one should not expect the existence of charged sectors if a mass is indeed generated as we expect. Very general arguments due to Swieca[11] and Buchholz and Fredenhagen[12] exclude the existence of charged superselection sectors in massive (continuum) theories.

Of course in the continuum limit the unwanted mass gap may disappear as it happens in the three-dimensional compact QED model without matter,[13] and charged superselection sectors may then suddenly appear. But none of this has been proven so far.

Finally our nonscreening results for plasma models almost certainly imply that in those models there is, strictly speaking, no Debye screening even for static electric sources. By Murphy's law the massless behavior of nonstatic correlation functions will undoubtedly pollute the static ones. In fact it is easy to write down perturbative contributions to the potential between static sources

$$V(x-y) \equiv -\log \left\langle \exp \left[ie \int_0^\beta A_0(x,\tau)\,d\tau \exp ie \int_0^\beta A_0(y,\tau)\,d\tau \right] \right\rangle$$

that show coupling to the massless modes. They are given by Feynman graphs like Fig. 1, where a wavy line denotes the propagator of the massless modes, a dotted line the (massive) propagator of the static mode, $\otimes$ the external sources, and a straight line the matter field propagator.

These graphs have three loops and eight internal vertices, so they carry a very small factor $\alpha^4\hbar^3$. For this reason they probably give an unobservably tiny contribution.

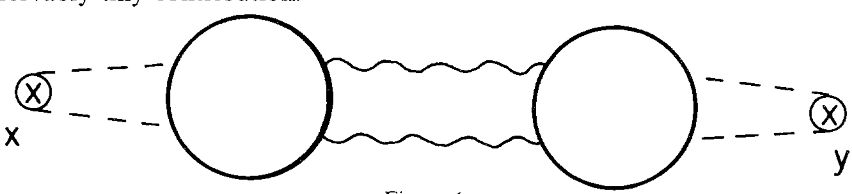

Figure 1

The situation is quite different for time dependent sources. Unfortunately the results of Section 5 deal with sources dependent on *imaginary time* and do not have a direct physical interpretation. But analytic continuation to real time should lead also to slow, powerlike decay of *real time* dependent correlations.

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NOTE ADDED IN PROOF

Our proof of Corollary 2.2 is incomplete because the two-point function for dA for free electromagnetism in dimension $d=2$ is a counterexample. Let $H(k)$ be the two-point function of dA in momentum space considered as a positive linear operator on L^2 2-forms. The claim is that continuity of $H(k)$ at $k=0$ implies $H(o)=0$ if $d>2$. Proof: Let $\hat{d}(k)$ be the exterior derivative in momentum space, then $\hat{d}(k)^* H(k) \hat{d}(k)=0$. By dividing by k^2 and taking the limit as $k\to 0$ we conclude that, for any u in $\mathbb{R}^d \wedge \mathbb{R}^d$ and any $e\in\mathbb{R}^d$, $(e \llcorner u,\ H(o)\, e \llcorner u)=0$. If $d\geqslant 3$ this implies $H(o)\geqslant 0$ by choosing $e=e_1\wedge e_2$ with e_1, e_2, e mutually perpendicular.

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